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The Existence and Local Behaviour of the Quadratic Function Approximation

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This paper analyses the local behaviour of the quadratic function approximation to a function which has a given power series expansion about the origin. It is shown that the quadratic Hermite–Padé form *always* defines a quadratic function and that this function is analytic in a neighbourhood of the origin. This result holds even if the origin is a critical point of the function (i.e., the discriminant has a zero at the origin). If the discriminant has multiple zeros the order of the approximation will be degraded but only to a limited extent. © 1990 Academic Press, Inc.

1. INTRODUCTION

This paper is concerned with the properties of the quadratic Hermite–Padé approximation. This approximation may be defined as follows (see, for example, Della Dora [3] or Baker and Lubinsky [2]).

Let $f(x)$ be a function, analytic in some neighbourhood of the origin, whose power series expansion about the origin is known. Let A_0, A_1, $A_2 \in \mathbf{Z}^+$ and $a_0(x)$, $a_1(x)$, $a_2(x)$ be polynomials in x with $\deg(a_i(x)) \leqslant A_i$, $i \in \{0, 1, 2\}$, such that

$$a_2(x)\, f(x)^2 + a_1(x)\, f(x) + a_0(x) = O(x^{A_0 + A_1 + A_2 + 2}). \tag{1}$$

Note that such $a_i(x)$, not all zero, must exist since (1) represents a homogeneous system of $A_0 + A_1 + A_2 + 2$ linear equations in the $A_0 + A_1 + A_2 + 3$ unknown coefficients of the $a_i(x)$. Then set

$$a_2(x)\, y(x)^2 + a_1(x)\, y(x) + a_0(x) = 0$$

and attempt to solve this equation for $y(x)$ in such a way that $y(x)$ approximates $f(x)$.

In the well-known case of Padé approximation (Baker [1]) the same procedure is followed for $a_1(x)\, f(x) + a_0(x) = O(x^{A_1 + A_0 + 1})$ which gives

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$y(x) = -a_0(x)/a_1(x)$. If $a_1(0) \neq 0$ (not a serious restriction) it then follows that $y(x) = f(x) + O(x^{A_0+A_1+1})$. However, in the quadratic case it is not obvious that $a_2(x)\, y(x)^2 + a_1(x)\, y(x) + a_0(x) = 0$ yields even an analytic approximation to $f(x)$, still less that it defines a function $y(x)$ such that $y(x) = f(x) + O(x^{A_0+A_1+A_2+2})$. The purpose of this paper is to show that an analogue of the Padé result is in fact true.

2. Notation

It will be assumed that

$$\sum_{j=0}^{2} a_j(x)\, f(x)^j = O(x^{N+2}),$$

where $N \geqslant \sum_j A_j$ and that $\sum_j |a_j(0)| \neq 0$. Note that if x^r is a common factor of the $a_j(x)$, $j \in \{0, 1, 2\}$ (r maximal) then

$$\sum_{j=0}^{2} \frac{a_j(x)}{x^r} f(x)^j = O(x^{N+2-r})$$

so that this second assumption is not a serious restriction.

The following notation will be used:

(i) An approximation derived from $\sum_{j=0}^{2} a_j(x)\, f(x)^j = O(x^{N+2})$ will be referred to as a (A_2, A_1, A_0) (quadratic) approximation to $f(x)$.

(ii) By $\sqrt{D(x)}$ we mean the principal square root of $D(x)$.

(iii) Let $D(x) = a_1(x)^2 - 4a_2(x)\, a_0(x)$. If $\sum_j a_j(x)\, y(x)^j = 0$ then $y(x) = (-a_1(x) \pm \sqrt{D(x)})/2a_2(x)$ and $\pm\sqrt{D(x)} = 2a_2(x)\, y(x) + a_1(x) = (\partial/\partial y)(\sum_j a_j(x)\, y(x)^j)$.

3. The Principal Results

The problem divides itself into two cases, the case $D(0) \neq 0$ and the case $D(0) = 0$.

3.1. *The Case* $D(0) \neq 0$

THEOREM 1. *If $D(0) \neq 0$ then there exists a unique function $y(x)$, analytic in a neighbourhood of the origin, satisfying $\sum_j a_j(x)\, y(x)^j = 0$ and $y(0) = f(0)$.*

Proof. The existence of a function $y(x)$, analytic about the origin, satisfying $\sum_j a_j(x)\, y(x)^j = 0$ follows from standard algebraic function theory (see, for example, Hille [4, Theorem 12.2.1]). However, in this special case it is easier to argue as follows.

Suppose $a_2(0) \neq 0$. The two possible expressions for $y(x)$ in a neighbourhood of the origin are given by

$$y(x) = \frac{-a_1(x) - \sqrt{D(x)}}{2a_2(x)}$$

or

$$y(x) = \frac{-a_1(x) + \sqrt{D(x)}}{2a_2(x)}.$$

Since $D(0) \neq 0$ these are both analytic in a neighbourhood of the origin. Exactly one of them satisfies $y(0) = f(0)$ because $a_2(0) f(0)^2 + a_1(0) f(0) + a_0(0) = 0$

$$\Rightarrow f(0) = \frac{-a_1(0) \pm \sqrt{D(0)}}{2a_2(0)}.$$

Suppose $a_2(0) = 0$. Then $a_1(0) \neq 0$ (again since $D(0) \neq 0$). Near the origin the two possible expressions for $y(x)$ can be written as

$$y(x) = \frac{-a_1(x) + a_1(x)\sqrt{1 - 4a_2(x)\, a_0(x)/a_1(x)^2}}{2a_2(x)} \tag{2}$$

or

$$y(x) = \frac{-a_1(x) - a_1(x)\sqrt{1 - 4a_2(x)\, a_0(x)/a_1(x)^2}}{2a_2(x)}. \tag{3}$$

The right hand side of (3) is unbounded as $x \to 0$ so we can exclude this possibility. Since $a_2(0) = 0$, close to the origin we can apply the binomial theorem to get from (2) the convergent power series (analytic in a neighbourhood of the origin) expression for $y(x)$:

$$\begin{aligned} y(x) &= \left(-a_1(x) + a_1(x)\left(1 + \sum_{i=1}^{\infty} \mu_i \left(\frac{4a_2(x)\, a_0(x)}{a_1(x)^2}\right)^i\right)\right) \Big/ 2a_2(x) \\ &= \sum_{i=1}^{\infty} \mu_i \left(\frac{4a_2(x)\, a_0(x)}{a_1(x)^2}\right)^{i-1} \frac{2a_0(x)}{a_1(x)}. \end{aligned}$$

Noting that $\mu_1 = -1/2$ it follows that

$$y(x) = \begin{cases} \dfrac{-a_1(x) + a_1(x)\sqrt{1 - 4a_2(x)\, a_0(x)/a_1(x)}}{2a_2(x)}, & x \neq 0 \\ -\dfrac{a_0(x)}{a_1(x)}, & x = 0 \end{cases}$$

is the only function, analytic in a neighbourhood of the origin, satisfying $\sum_j a_j(x)\, y(x)^j = 0$ with $y(0) = f(0)$. ∎

THEOREM 2. *If $D(0) \neq 0$ then there exists a unique function $y(x)$, analytic in a neighbourhood of the origin, satisfying $\sum_j a_j(x)\, y(x)^j = 0$ such that*

$$y(x) = f(x) + O(x^{N+2}).$$

Proof. Note that

$$\frac{d^i}{dx^i}\left[\sum_j a_j(x)\, y(x)^j\right]\Bigg|_0 = 0 = \frac{d^i}{dx^i}\left[\sum_j a_j(x)\, f(x)^j\right]\Bigg|_0, \qquad i \in \{0, ..., N+1\}. \quad (4)$$

For $i = 1$

$$\left[\frac{\partial}{\partial y}\left(\sum a_j(x)\, y(x)^j\right)\frac{dy}{dx} + \left(\sum \frac{d}{dx}(a_j(x))\, y(x)^j\right)\right]\Bigg|_0 = 0$$

$$\left[\frac{\partial}{\partial f}\left(\sum a_j(x)\, f(x)^j\right)\frac{df}{dx} + \left(\sum \frac{d}{dx}(a_j(x))\, f(x)^j\right)\right]\Bigg|_0 = 0.$$

Differentiating again $(i = 2)$ gives

$$\left[\frac{\partial}{\partial y}\left(\sum a_j(x)\, y(x)^j\right)\frac{d^2y}{dx^2} + \frac{d}{dx}\left(\frac{\partial}{\partial y}\left(\sum a_j(x)\, y(x)^j\right)\right)\frac{dy}{dx} + \frac{d}{dx}\left(\sum \frac{d}{dx}(a_j(x))\, y(x)^j\right)\right]\Bigg|_0 = 0$$

$$\left[\frac{\partial}{\partial f}\left(\sum a_j(x)\, f(x)^j\right)\frac{d^2f}{dx^2} + \frac{d}{dx}\left(\frac{\partial}{\partial f}\left(\sum a_j(x)\, f(x)^j\right)\right)\frac{df}{dx} + \frac{d}{dx}\left(\sum \frac{d}{dx}(a_j(x))\, f(x)^j\right)\right]\Bigg|_0 = 0.$$

In a general, more compact form we have

$$\left[\frac{\partial}{\partial y}\left(\sum a_j(x)\, y(x)^j\right)\frac{d^i y}{dx^i}+z_i\right]\bigg|_0=0=\left[\frac{\partial}{\partial f}\left(\sum a_j(x)\, f(x)^j\right)\frac{d^i f}{dx^i}+Z_i\right]\bigg|_0,$$
$$i\in\{1, ..., N+1\}, \qquad (5)$$

where

$$z_1=\sum\frac{d}{dx}(a_j(x))\; y(x)^j,$$

$$z_{i+1}=z_{i+1}\left(x, y(x), \frac{dy}{dx}, ..., \frac{d^i y}{dx^i}\right)=\frac{dz_i}{dx}+\frac{d^i y}{dx^i}\frac{d}{dx}\left(\frac{\partial}{\partial y}\sum a_j(x)\, y(x)^j\right),$$

$$Z_{i+1}=z_{i+1}\left(x, f(x), \frac{df}{dx}, ..., \frac{d^i f}{dx^i}\right).$$

Now, taking the unique $y(x)$ from Theorem 1 it is seen that since $\pm\sqrt{D(0)}=(\partial/\partial y)(\sum a_j(x)\, y(x)^j)|_0\neq 0$, Eq. (5) with $i=1$ gives $(df/dx)|_0=(dy/dx)|_0$, which with $i=2$ gives $(d^2f/dx^2)|_0=(d^2y/dx^2)|_0$. It follows that

$$\frac{d^i f}{dx^i}\bigg|_0=\frac{d^i y}{dx^i}\bigg|_0, \qquad i\in\{0, ..., N+1\},$$

i.e.,

$$y(x)=f(x)+O(x^{N+2}). \qquad \blacksquare$$

3.2. *The Case* $D(0)=0$

We now investigate the case $D(0)=0$. This implies that $a_2(0)\neq 0$ (since if $D(0)=a_1(0)^2-4a_2(0)\, a_0(0)=0$ and $a_2(0)=0$ then $a_1(0)=0$, which with $a_2(0)\, f(0)^2+a_1(0)\, f(0)+a_0(0)=0$ gives $a_0(0)=0$; this contradicts the assumption that $\sum_j |a_j(0)|\neq 0$).

Bearing in mind that $y(x)=(-a_1(x)\pm\sqrt{D(x)})/2a_2(x)$ and $\sqrt{D(x)}$ is not now analytic at the origin this case does not seem well-behaved, but such is not the case. Certainly if $D(x)$ has a root of odd multiplicity at the origin then any $y(x)$ satisfying $\sum_j a_j(x)\, y(x)^j=0$ is not analytic at the origin since

$$\lim_{t\to 0}\frac{d^{r+1}}{dx^{r+1}}\frac{\sqrt{xg(x)}}{a(x)}x^r\bigg|_t\to\infty \qquad (g(0)\neq 0).$$

[Take, for example, $x\sqrt{x}$. Then $(d^2/dx^2)(x\sqrt{x})=3/(4\sqrt{x})$. This generalises easily (using the Leibnitz rule) to the above.] However, this case never occurs in practice.

First, it is necessary to treat two special cases:

(i) Suppose $a_0(x) \equiv 0$.

Then

$$a_2(x)\, f(x)^2 + a_1(x)\, f(x) = O(x^{N+2})$$
$$\Rightarrow (a_2(x)\, f(x) + a_1(x))\, f(x) = O(x^{N+2})$$

so that

$$\left.\begin{array}{ll} -a_1(x)/a_2(x) & = f(x) + O(x^R) \\ 0 & = f(x) + O(x^S) \end{array}\right\}, \qquad \text{where} \quad R + S = N + 2.$$

Choosing

$$\begin{cases} y(x) = -\dfrac{a_1(x)}{a_2(x)} & \text{if} \quad R > S \\ y(x) = 0 & \text{otherwise} \end{cases}$$

gives $y(x)$ such that

$$\sum_i a_i(x)\, y(x)^i = 0$$

and $y(x) = f(x) + O(x^{\max\{R,S\}})$. (Clearly $\max\{R, S\} \geqslant N/2 + 1$.)

(ii) Suppose $D(x) \equiv 0$. Then

$$a_2(x)\, f(x)^2 + a_1(x)\, f(x) + a_0(x) = O(x^{N+2})$$
$$\Rightarrow (2a_2(x)\, f(x) + a_1(x))^2 = 4a_2(x)\, O(x^{N+2}) = O(x^{N+2})$$
$$\Rightarrow y(x) = -\frac{a_1(x)}{2a_2(x)} = f(x) + O(x^T), \qquad T = \min\left\{t \in \mathbf{N}\colon t \geqslant \frac{N}{2} + 1\right\}$$

and $\sum_i a_i(x)\, y(x)^i = 0$.

It will be assumed for the remainder of this section that neither $D(x) \equiv 0$ nor $a_0(x) \equiv 0$.

THEOREM 3. *Let* $C_i = \deg(a_i(x))$. *Then* $D(x)$ *never has a root of multiplicity greater than* $\sum_i C_i$ *at the origin.*

Proof. Let $\sum_i C_i = M$ and suppose $D(x) = x^{M+1} p_r(x)$, $p_r(x)$ a polynomial of degree r. Since $a_2(x)$, $a_0(x) \not\equiv 0$, then

$$a_1(x)^2 = x^{M+1} p_r(x) + q_s(x), \tag{6}$$

where $q_s(x)$ is a (nonzero) polynomial of degree s. We must have $M+1+r=2C_1$ (since $C_2+C_0 \leqslant M < M+1$) so that $q_s(x)=4a_0(x)\,a_2(x)$. Also $s+C_1=C_0+C_2+C_1<M+1=2C_1-r \Rightarrow s+r<C_1$. Differentiating (6)

$$\begin{aligned} 2a_1(x)\,a_1'(x) &= x^M((M+1)\,p_r(x)+xp_r'(x))+q_s'(x) \\ \Rightarrow 2xa_1(x)\,a_1'(x) &= x^{M+1}((M+1)\,p_r(x)+xp_r'(x))+xq_s'(x) \\ &= x^{M+1}\bar{p}_r(x)+\bar{q}_s(x), \end{aligned} \tag{7}$$

where

$$\bar{p}_r(x)=(M+1)\,p_r(x)+xp_r'(x) \qquad \text{(degree } r\text{)}$$

$$\bar{q}_s(x)=xq_s'(x) \qquad \text{(degree } s\text{)}.$$

From (6) and (7) (eliminating the term in x^{M+1})

$$a_1(x)(\bar{p}_r(x)\,a_1(x)-2p_r(x)\,xa_1'(x))=\begin{vmatrix} q_s(x) & p_r(x) \\ \bar{q}_s(x) & \bar{p}_r(x) \end{vmatrix}. \tag{8}$$

The left-hand side of (8) either has degree $\geqslant C_1$ or is identically zero, while the right-hand side has degree $\leqslant s+r<C_1$. It follows that

$$\bar{p}_r(x)\,a_1(x)-2p_r(x)\,xa_1'(x)=0=q_s(x)\,\bar{p}_r(x)-p_r(x)\,\bar{q}_s(x).$$

Hence

$$\frac{a_1'(x)}{a_1(x)}=\frac{\bar{p}_r(x)}{2xp_r(x)}=\frac{\bar{q}_s(x)}{2xq_s(x)}=\frac{q_s'(x)}{2q_s(x)}$$

and integrating gives

$$a_1(x)=k\sqrt{q_s(x)}, \qquad k \in \mathbf{R}.$$

But $\deg\sqrt{q_s(x)}=s/2<C_1$ so the result is proved by contradiction. ∎

THEOREM 4. *$D(x)$ never has a root of odd multiplicity at the origin.*

Proof. Suppose $D(x)=x^{2s+1}g(x)$, $g(0)\neq 0$. By Theorem 3 it can be assumed that $2s+1<N+1$. Then

$$\frac{d^{2s+1}}{dx^{2s+1}}D(x)|_0\neq 0, \tag{9}$$

$$\frac{d^i}{dx^i}D(x)\bigg|_0=0, \qquad i\in\{0, ..., 2s\}. \tag{10}$$

Let $G(x) = (2a_2(x) f(x) + a_1(x))$. Then

$$a_2(x) f(x)^2 + a_1(x) f(x) + a_0(x) = O(x^{N+2})$$
$$\Rightarrow G(x)^2 - D(x) = 4a_2(x)\, O(x^{N+2}) = O(x^{N+2}). \tag{11}$$

From (10) and (11)

$$\left.\frac{d^i}{dx^i} G(x)^2\right|_0 = 0, \qquad i \in \{0, ..., 2s\} \tag{12}$$
$$\Rightarrow \sum_{j=0}^{i} \binom{i}{j} \left.\frac{d^j}{dx^j} G(x) \frac{d^{i-j}}{dx^{i-j}} G(x)\right|_0 = 0, \qquad i \in \{0, ..., 2s\}$$
$$\Rightarrow \left.\frac{d^i}{dx^i} G(x)\right|_0 = 0, \qquad i \in \{0, ..., s\}.$$

[Expanding the first few equations,

$$G(x)^2|_0 = 0 \Rightarrow G(x)|_0 = 0$$
$$\left.\frac{d}{dx} G(x)^2\right|_0 = 0 \Rightarrow [G(x)\, G'(x) + G'(x)\, G(x)]|_0 = 0$$
$$\left.\frac{d^2}{dx^2} G(x)^2\right|_0 = 0 \Rightarrow [G(x)\, G''(x) + 2G'(x)^2 + G''(x)\, G(x)]|_0 = 0$$
$$\Rightarrow G'(x)|_0 = 0]$$
$$\Rightarrow \left.\frac{d^{2s+1}}{dx^{2s+1}} G(x)^2\right|_0 = 0$$
$$\Rightarrow \left.\frac{d^{2s+1}}{dx^{2s+1}} D(x)\right|_0 = 0.$$

Hence the result is shown by contradiction. ∎

THEOREM 5. *If* $D(x) = x^{2s} g(x)$, $g(0) \neq 0$, $2s < N+1$ *then either*

$$y(x) = \frac{-a_1(x) + x^s \sqrt{g(x)}}{2a_2(x)}$$

or

$$y(x) = \frac{-a_1(x) - x^s \sqrt{g(x)}}{2a_2(x)}$$

satisfies

$$\sum a_j(x)\, y(x)^j = 0 \qquad \textit{and} \qquad y(x) = f(x) + O(x^{N+2-s}).$$

Proof. Let $h(x) = x^s\sqrt{g(x)}$. Then $G(x)^2 - h(x)^2 = O(x^{N+2})$ ($G(x)$ as defined in the proof of Theorem 4)

$$\Rightarrow \frac{d^i}{dx^i} G(x)^2\bigg|_0 = \frac{d^i}{dx^i} h(x)^2\bigg|_0, \qquad i \in \{0, ..., N+1\}. \tag{13}$$

Also $(d^i/dx^i)\, G(x)^2|_0 = 0 = (d^i/dx^i)\, h(x)^2|_0$, $i \in \{0, ..., 2s-1\}$, so $(d^i/dx^i)\, G(x)|_0 = 0 = (d^i/dx^i)\, h(x)|_0$, $i \in \{0, ..., s-1\}$ (using ideas in the proof of Theorem 4). But $(d^{2s}/dx^{2s})\, G(x)^2|_0 = (d^{2s}/dx^{2s})\, h(x)^2|_0 \neq 0$

$$\Rightarrow \left(\frac{d^s}{dx^s} G(x)\bigg|_0\right)^2 = \left(\frac{d^s}{dx^s} h(x)\bigg|_0\right)^2 \neq 0.$$

Now choose $t(x) = h(x)$ or $t(x) = -h(x)$ so that $(d^s/dx^s)\, G(x)|_0 = (d^s/dx^s)\, t(x)|_0$. Then Eq. (13) with $i = 2s+1$ gives

$$\frac{d^{2s+1}}{dx^{2s+1}} G(x)^2\bigg|_0 = \frac{d^{2s+1}}{dx^{2s+1}}\, t(x)^2\bigg|_0$$

$$\Rightarrow \left(\frac{d^s}{dx^s} G(x) \frac{d^{s+1}}{dx^{s+1}} G(x)\right)\bigg|_0 = \left(\frac{d^s}{dx^s} t(x) \frac{d^{s+1}}{dx^{s+1}} t(x)\right)\bigg|_0.$$

We progress in this way up to $i = N+1$ (note that if $2s = N+1$ this procedure is not required).

It follows that

$$\frac{d^i}{dx^i} G(x)\bigg|_0 = \frac{d^i}{dx^i} t(x)\bigg|_0, \qquad i \in \{0, ..., N+1-s\},$$

i.e., $2a_2(x)\, f(x) + a_1(x) = t(x) + O(x^{N+2-s})$. Since $a_2(0) \neq 0$, defining

$$y(x) = -\frac{a_1(x) - t(x)}{2a_2(x)}$$

gives $y(x) = f(x) + O(x^{N+2-s})$. ∎

4. Illustrative Examples

This paper is an attempt to answer many of the practical questions which arise when one actually tries to compute the quadratic approximation to some function. The seemingly exceptional cases covered by the previous theorems do frequently occur as is shown below.

EXAMPLE 1. Let $f(x) = \log(1+x)$. Then $xf(x)^2 + (-6x-12)f(x) + 12x = O(x^5)$. Also

$$y(x) = \frac{6x+12-\sqrt{(6x+12)^2-48x^2}}{2x}$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{29x^5}{144} + \cdots,$$

i.e., $y(x) = f(x) + O(x^5)$ (cf. the case $a_2(0) = 0$ in the proof of Theorem 1).

EXAMPLE 2. Let $f(x) = \log(1+x)$. Then $(x^2-6x-6)f(x)^2 + (-9x^2-18x)f(x) + 24x^2 = O(x^8)$. Note that $D(x) = -15x^4 + 900x^3 + 900x^2$.

Also

$$y(x) = \frac{9x^2+18x-x\sqrt{-15x^2+900x+900}}{2(x^2-6x-6)}$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{1543x^7}{10800} + \cdots,$$

i.e., $y(x) = f(x) + O(x^7)$ as predicted by Theorem 5.

EXAMPLE 3. Let

$$f(x) = 2x^5 + \frac{x^6}{2} - \frac{x^7}{8} + \frac{x^8}{16} - \frac{5x^9}{128} + \frac{7x^{10}}{256} - \frac{21x^{11}}{1024} + \frac{33x^{12}}{2048} + O(x^{13}).$$

Then $f(x)^2 - 2x^5 f(x) - x^{11} = O(x^{18})$. Note that $D(x) = 4x^{11} + 4x^{10}$. Also

$$y(x) = x^5 + x^5\sqrt{x+1}$$
$$= 2x^5 + \frac{x^6}{2} - \frac{x^7}{8} + \frac{x^8}{16} - \frac{5x^9}{128} + \frac{7x^{10}}{256}$$
$$- \frac{21x^{11}}{1024} + \frac{33x^{12}}{2048} - \frac{429x^{13}}{32768} + \cdots,$$

i.e., $y(x) = f(x) + O(x^{13})$ as predicted by Theorem 5.

5. Sequences of Quadratic Hermite–Padé Approximations

In this section it is shown that almost any "increasing" sequence of Hermite–Padé forms yields a sequence of approximations with increasing order of accuracy.

Let $R = \max\{r \in \mathbf{N}\colon f(x) = O(x^r)\}$ and let $\{(A_2^i, A_1^i, A_0^i), A_j^i, i \in \mathbf{N}\}$ be a sequence of number triples satisfying:

(i) $A_2^i + A_0^i + 1 \geqslant R$

(ii) $(A_0^i + A_1^i)/2 \geqslant R$

(iii) $\lim_{i \to \infty} \sum_j A_j^i = \infty$.

(i) and (ii) are minor conditions, avoiding undue degeneracy of corresponding Hermite–Padé forms (see the following lemma), while (iii) ensures that such a sequence of forms has increasing order.

Let $\{(a_2^i(x), a_1^i(x), a_0^i(x))\colon i \in \mathbf{N}\}$ be a sequence of polynomial coefficients of Hermite–Padé forms corresponding to $\{(A_2^i, A_1^i, A_0^i)\colon i \in \mathbf{N}\}$.

Lemma 6. *$\forall i \in \mathbf{N}$ at least two of the coefficients $\{a_2^i(x), a_1^i(x), a_0^i(x)\}$ are not identically zero.*

Proof. If $a_2^i(x) \equiv 0 \equiv a_1^i(x)$ then $a_0^i(x) = O(x^{A_0^i + 2})$ which is impossible. If $a_2^i(x) \equiv 0 \equiv a_0^i(x)$ then

$$a_1^i(x)\, f(x) = O(x^{A_2^i + A_1^i + A_0^i + 2})$$
$$\Rightarrow f(x) = O(x^{A_2^i + A_0^i + 2})$$

which contradicts (i) above.

If $a_1^i(x) \equiv 0 \equiv a_0^i(x)$ then

$$a_2^i(x)\, f(x)^2 = O(x^{A_2^i + A_1^i + A_0^i + 2})$$
$$\Rightarrow f(x)^2 = O(x^{A_1^i + A_0^i + 2})$$
$$\Rightarrow f(x) = O(x^{(A_1^i + A_0^i + 2)/2})$$

which contradicts (ii) above. ∎

Theorem 7. *Let $\{[A_2^i, A_1^i, A_0^i]\colon i \in \mathbf{N}\}$ be a sequence of Hermite–Padé forms with the above properties. Then each $[A_2^i, A_1^i, A_0^i]$ gives a quadratic approximation to $f(x)$ such that*

(i) $y_i(x) = f(x) + O(x^{n_i})$

(ii) $\lim_{i \to \infty} n_i = \infty$.

Proof. Let $i \in \mathbf{N}$, $r_i = \max\{r \in \mathbf{N}: x^r \mid a_j^i(x),\ \forall j \in \{0, 1, 2\}\}$ and $a_j^i(x) = x^{r_i} g_j^i(x)$, $j \in \{0, 1, 2\}$. Then $\sum_j g_j^i(x) f(x)^j = O(x^{N_i+2-r_i})$ (where $N_i = \sum_j A_j^i$).

Using the previous results, there exists $y_i(x)$ a quadratic (possibly rational or polynomial) approximation such that

$$y_i(x) = f(x) + O(x^{(N_i+2-r_i)/2}).$$

Since $x^{r_i} \mid a_j^i(x)$, $\forall j \in \{0, 1, 2\}$, and at least two of the $a_j^i(x)$ are not identically zero then

$$N_i \geqslant 2r_i.$$

Consider the sequence $\{N_i - r_i : i \in \mathbf{N}\}$. Since $\lim_{i \to \infty} N_i = \infty$ and $N_i - r_i \geqslant N_i - N_i/2 = N_i/2$ then $\lim_{i \to \infty}(N_i - r_i) = \infty$. Letting $n_i = (N_i - r_i + 2)/2$ it follows that $\lim_{i \to \infty} n_i = \infty$ and the result is proven. ∎

6. Conclusion

These results show that given $a_2(x) f(x)^2 + a_1(x) f(x) + a_0(x) = O(x^{N+2})$, $\sum_{i=0}^{2} |a_i(0)| \neq 0$ then we can always find $y(x)$ such that $\sum_{i=0}^{2} a_i(x)\, y(x)^i = 0$ and $y(x) = f(x) + O(x^K)$ where K is, at worst, $N/2 + 1$. This is summarised in Table I.

It then follows that most increasing sequences of quadratic Hermite Padé forms give sequences of quadratic approximinations of increasing order of accuracy. It is hoped that this work will be useful in attempting to extend convergence results such as that given by Baker and Lubinsky [2] to the so-called "non-normal" case in quadratic approximation.

TABLE I

Case	K
$D(0) \neq 0$	$N+2$
$D(x) = x^{2s} g(x)$ where $g(0) \neq 0$, $2s < N+1$, $a_0(x) \not\equiv 0$	$N+2-s$
$D(0) = 0$ and $a_0(x) \equiv 0$	$\min\{k \in \mathbf{N}: k \geqslant (N/2)+1\}$
$D(x) \equiv 0$	$\min\{k \in \mathbf{N}: k \geqslant (N/2)+1\}$

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